

Renormalization of minimally doubled fermions

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Abstract

We investigate the renormalization properties of minimally doubled fermions, at one loop in perturbation theory. Our study is based on the two particular realizations of Boriçi-Creutz and Karsten-Wilczek. A common feature of both formulations is the breaking of hyper-cubic symmetry, which requires that the lattice actions are supplemented by suitable counterterms. We show that three counterterms are required in each case and determine their coefficients to one loop in perturbation theory. For both actions we compute the vacuum polarization of the gluon. It is shown that no power divergences appear and that all contributions which arise from the breaking of Lorentz symmetry are cancelled by the counterterms. We also derive the conserved vector and axial-vector currents for Karsten-Wilczek fermions. Like in the case of the previously studied Boriçi-Creutz action, one obtains simple expressions, involving only nearest-neighbour sites. We suggest methods how to fix the coefficients of the counterterms non-perturbatively and discuss the implications of our findings for practical simulations.

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1 Introduction

Minimally doubled fermions [1–14] preserve an exact chiral symmetry for a degenerate doublet of quarks, thereby realizing the minimal doubling of fermion species allowed by the Nielsen-Ninomiya theorem [15–17]. At the same time they also remain strictly local, and may thus be regarded as a cost-effective realization of chiral symmetry at non-zero lattice spacing, which is particularly suited for simulating a degenerate light doublet of up and down quarks.

In a previous article [11] we began an investigation into the renormalization properties of a particular realization of minimally doubled fermions, described by the Boriçi-Creutz action [3, 4, 9, 10], based on perturbation theory at one loop. In this paper¹ we present a similar analysis for another member of this class of fermions, proposed a long time ago by Wilczek [2], following previous work of Karsten [1]. Here we take the opportunity to revise and sharpen some of the conclusions of our earlier article [11], by interpreting our results in a more general field-theoretical framework. In particular, what we interpreted in ref. [11] as a shift in the quark’s four-momentum can, in fact, be absorbed by adding appropriate counterterms to the lattice actions, so that any momentum shift disappears in the fully renormalized theory. The main focus of the present article is on the one-loop structure of minimally doubled fermions, taking the Boriçi-Creutz and Karsten-Wilczek actions as particular examples. We identify the possible counterterms which can arise and study the consequences that can be inferred from their presence.

Our findings can be summarized by the following: a consistent renormalized theory for minimally doubled fermions can be constructed by fixing the coefficients of three counterterms in the action which are allowed by the symmetries. In this work we determine these coefficients in perturbation theory, and discuss possible renormalization conditions to fix their values non-perturbatively. One of the principal results, namely the full expressions for the renormalized actions including all counterterms, can be found in eqs. (75) and (76). We also present the computation of the vacuum polarization for both particular formulations. Our calculation demonstrates that radiative corrections at one loop do not introduce new divergences for this quantity.

This article is organized as follows. After defining the two actions and the corresponding propagators and vertices in Section 2, we introduce in Section 3 the necessary counterterms which render the theories consistent under renormalization. With these tools in hand we fix the coefficients of the counterterms which appear in the quark action in Section 4, compute the matrix elements of quark bilinears and derive the conserved currents in Section 5 and also show that their renormalization constant is one. In Section 6 we present the calculation of the vacuum polarization of the gluon, while in Section 7 we discuss the implications of our findings for the practical implementation of minimally doubled fermions in numerical simulations. Finally, we make some concluding remarks.

2 Actions, propagators and vertices

In order to make this article self-contained, we recall here the basic definitions for Boriçi-Creutz fermions [3, 4, 9, 10] and also introduce the Karsten-Wilczek action [1, 2]. In position space the lattice action of Boriçi-Creutz fermions reads

$$S_{\text{BC}}^f = a^4 \sum_x \left[\frac{1}{2a} \sum_{\mu=1}^4 \left[\bar{\psi}(x) (\gamma_\mu + i\gamma'_\mu) U_\mu(x) \psi(x + a\hat{\mu}) \right. \right.$$

¹Preliminary results were reported in [12].

$$- \bar{\psi}(x + a\hat{\mu}) (\gamma_{\mu} - i\gamma'_{\mu}) U_{\mu}^{\dagger}(x) \psi(x) \Big] + \bar{\psi}(x) \left(m_0 - \frac{2i\Gamma}{a} \right) \psi(x) \Big], \quad (1)$$

where the matrices Γ and γ'_{μ} are defined by

$$\Gamma = \frac{1}{2} \sum_{\mu=1}^4 \gamma_{\mu}, \quad \gamma'_{\mu} = \Gamma \gamma_{\mu} \Gamma = \Gamma - \gamma_{\mu}, \quad (2)$$

with $\Gamma^2 = 1$. The Karsten-Wilczek action reads²

$$S_{\text{KW}}^f = a^4 \sum_x \left[\frac{1}{2a} \sum_{\mu=1}^4 \left[\bar{\psi}(x) (\gamma_{\mu} - i\gamma_4 (1 - \delta_{\mu 4})) U_{\mu}(x) \psi(x + a\hat{\mu}) \right. \right. \\ \left. \left. - \bar{\psi}(x + a\hat{\mu}) (\gamma_{\mu} + i\gamma_4 (1 - \delta_{\mu 4})) U_{\mu}^{\dagger}(x) \psi(x) \right] + \bar{\psi}(x) \left(m_0 + \frac{3i\gamma_4}{a} \right) \psi(x) \right]. \quad (3)$$

We remind the reader that one Dirac spinor, $\psi(x)$, in these expressions describes a degenerate doublet of quarks. In momentum space the free Dirac operator of Boriçi-Creutz fermions takes the form

$$\mathcal{D}_{\text{BC}}(p) = D(p) + \bar{D}(p) - \frac{2i\Gamma}{a} + m_0, \quad (4)$$

with $D(p)$ and $\bar{D}(p)$ given by

$$D(p) = \frac{i}{a} \sum_{\mu=1}^4 (\gamma_{\mu} \sin ap_{\mu}), \quad \bar{D}(p) = \frac{i}{a} \sum_{\mu=1}^4 (\gamma'_{\mu} \cos ap_{\mu}). \quad (5)$$

This action has two doublers, corresponding to the two zeros (Fermi points) at $ap_1 = (0, 0, 0, 0)$ and $ap_2 = (\pi/2, \pi/2, \pi/2, \pi/2)$. For Karsten-Wilczek fermions the free Dirac operator reads

$$\mathcal{D}_{\text{KW}}(p) = \frac{i}{a} \sum_{\mu=1}^4 \gamma_{\mu} \sin ap_{\mu} + \frac{i}{a} \gamma_4 \sum_{k=1}^3 (1 - \cos ap_k) + m_0 = D(p) + \frac{i}{a} \gamma_4 \sum_{k=1}^3 (1 - \cos ap_k) + m_0. \quad (6)$$

In the latter case the term proportional to γ_4 , which is chirally invariant since it anticommutes with γ_5 , removes 14 of the doublers of the naive fermion action $D(p)$, and only the doubler whose pole lies entirely in the temporal direction survives. The Dirac operator $\mathcal{D}_{\text{KW}}(p)$ of the Karsten-Wilczek action exhibits then only two Fermi points, which are located at $ap_1 = (0, 0, 0, 0)$ and $ap_2 = (0, 0, 0, \pi)$ and describe – like for Boriçi-Creutz fermions – two degenerate fermion species of opposite chirality.

In principle, the term proportional to γ_4 can be multiplied by some coefficient, λ , without spoiling the chiral symmetry of Karsten-Wilczek fermions, or modifying the minimal number of doublers. However, it is conceivable that for general values of λ the transfer matrix does not exist or that the theory presents some other kind of problem. In this work we stick to the case $\lambda = 1$, which is the common choice in the literature.³

The two actions described here and investigated at length in this article represent two particular realizations of minimally doubled fermions, which respect chiral symmetry at any finite lattice spacing, but are no longer symmetric under the full hyper-cubic group. The

²In this article we will denote 4-vectors with Greek indices and spatial 3-vectors with Latin indices. The temporal component is $\mu = 4$.

³This coefficient λ is in many ways similar to the Wilson parameter of Wilson fermions, which is commonly set to one in simulations.

Boriçi-Creutz action is only symmetric with respect to a subgroup which preserves the hyper-cubic positive major diagonal. The term proportional to γ_4 in the Dirac operator of Karsten-Wilczek fermions selects, instead, a different particular direction in Euclidean space, the temporal axis.

The breaking of hyper-cubic symmetry allows for mixing with operators of various dimensions. In particular, mixing with lower-dimensional operators may occur, which implies the appearance of power-divergent coefficients proportional to $1/a^n$. In the following we will show that for these actions the breaking of hyper-cubic symmetry indeed generates linearly divergent counterterms, whose coefficients need to be determined via some physical condition.

We will see in the remainder of this article that in several instances the same reasoning applies to both actions considered. However, we would like to stress here that, despite many similarities, these two realizations of minimally doubled fermions are not equivalent. The ways in which the two actions are constructed, and the mechanisms for the removal of doublers, are indeed qualitatively different.⁴

Although the distance between the two Fermi points p_1 and p_2 is the same in each of the two actions considered (i.e., $p_2^2 - p_1^2 = \pi^2/a^2$), these two actions cannot in fact be transformed into each other by a 4-dimensional rotation. On the other hand, in this article we will show that many results turn out to be qualitatively similar, owing to the common feature of the breaking of the hyper-cubic symmetry by a fixed direction in four-dimensional space (determined by the two Fermi points).

It is easy to verify that both actions satisfy γ_5 -hermiticity, with all the advantages and simplifications that this property implies, especially in numerical simulations. Another interesting observation is that the Dirac operators can be written as

$$D_{\text{BC}}^f = \frac{1}{2} \left\{ \sum_{\mu=1}^4 \gamma_{\mu} (\nabla_{\mu} + \nabla_{\mu}^*) + ia \sum_{\mu=1}^4 \gamma'_{\mu} \nabla_{\mu}^* \nabla_{\mu} \right\}, \quad (7)$$

$$D_{\text{KW}}^f = \frac{1}{2} \left\{ \sum_{\mu=1}^4 \gamma_{\mu} (\nabla_{\mu} + \nabla_{\mu}^*) - ia \gamma_4 \sum_{k=1}^3 \nabla_k^* \nabla_k \right\}, \quad (8)$$

where $\nabla_{\mu} \psi(x) = \frac{1}{a} [U_{\mu}(x) \psi(x + a\hat{\mu}) - \psi(x)]$ is the nearest-neighbour forward covariant derivative, and ∇_{μ}^* the corresponding backward one. Thus, it becomes apparent that the two realizations of minimally doubled fermions bear a close formal resemblance to Wilson fermions, i.e.

$$D_{\text{Wilson}}^f = \frac{1}{2} \left\{ \sum_{\mu=1}^4 \gamma_{\mu} (\nabla_{\mu} + \nabla_{\mu}^*) - ar \sum_{\mu=1}^4 \nabla_{\mu}^* \nabla_{\mu} \right\}. \quad (9)$$

Moreover, this demonstrates the presence of dimension-five operators in all three cases. For Wilson fermions the dimension-five operator breaks chiral symmetry, while for minimally doubled fermions chiral symmetry is preserved at the expense of introducing operators which break hyper-cubic symmetry.

For the derivation of the propagators and vertices of Boriçi-Creutz fermions we refer

⁴There is some freedom in constructing actions of the Boriçi-Creutz or Karsten-Wilczek type. For instance, in the latter case one can replace $(1 - \cos ap_k)$ with other trigonometric functions without spoiling the basic properties of this kind of action, especially the fact that there are two Fermi points exactly located at $ap_1 = (0, 0, 0, 0)$ and $ap_2 = (0, 0, 0, \pi)$.

to [11]. Here we only remind that the Boriçi-Creutz quark propagator can be written as

$$S(p) = a \frac{-i \sum_{\mu} \left[\gamma_{\mu} \sin ap_{\mu} - 2 \gamma'_{\mu} \sin^2 ap_{\mu}/2 \right] + am_0}{4 \sum_{\mu} \left[\sin^2 ap_{\mu}/2 + \sin ap_{\mu} \left(\sin^2 ap_{\mu}/2 - \frac{1}{2} \sum_{\nu} \sin^2 ap_{\nu}/2 \right) \right] + (am_0)^2}, \quad (10)$$

the quark-quark-gluon vertex is given by

$$V_1(p_1, p_2) = -ig_0 \left(\gamma_{\mu} \cos \frac{a(p_1 + p_2)_{\mu}}{2} - \gamma'_{\mu} \sin \frac{a(p_1 + p_2)_{\mu}}{2} \right) \quad (11)$$

(where p_1 and p_2 are the incoming and outgoing quark momenta at the vertex), and the quark-quark-gluon-gluon vertex is

$$V_2(p_1, p_2) = \frac{1}{2} iag_0^2 \left(\gamma_{\mu} \sin \frac{a(p_1 + p_2)_{\mu}}{2} + \gamma'_{\mu} \cos \frac{a(p_1 + p_2)_{\mu}}{2} \right). \quad (12)$$

The fermion propagator of Karsten-Wilczek fermions is obtained by inverting the Dirac operator of eq. (6), and is given by

$$S(p) = a \frac{-i \sum_{\mu=1}^4 \gamma_{\mu} \sin ap_{\mu} - 2i \gamma_4 \sum_{k=1}^3 \sin^2 ap_k/2 + am_0}{\sum_{\mu=1}^4 \sin^2 ap_{\mu} + 4 \sin ap_4 \sum_{k=1}^3 \sin^2 ap_k/2 + 4 \left(\sum_{k=1}^3 \sin^2 ap_k/2 \right) \left(\sum_{l=1}^3 \sin^2 ap_l/2 \right) + (am_0)^2}. \quad (13)$$

As in the case of Boriçi-Creutz fermions, but unlike many standard fermionic discretizations, we find that the denominator of this propagator does not possess a simple behavior under all momentum inversions (in this case of the fourth direction, so that it amounts to a time reversal).

After making the substitution $ap_4 \rightarrow \pi + ap_4$, one obtains the propagator for the fermionic mode associated with the other Fermi point, the one at $ap = (0, 0, 0, \pi)$:

$$S(p) = a \frac{i \sum_{\mu=1}^4 \gamma'_{\mu} \sin ap_{\mu} - 2i \gamma'_4 \sum_{k=1}^3 \sin^2 ap_k/2 + am_0}{\sum_{\mu=1}^4 \sin^2 ap_{\mu} - 4 \sin ap_4 \sum_{k=1}^3 \sin^2 ap_k/2 + 4 \left(\sum_{k=1}^3 \sin^2 ap_k/2 \right) \left(\sum_{l=1}^3 \sin^2 ap_l/2 \right) + (am_0)^2}, \quad (14)$$

where we have also introduced a new set of Dirac matrices, $\gamma'_k = -\gamma_k$ and $\gamma'_4 = \gamma_4$. If we now invert the direction of the four-momentum p_{μ} in this expression, the propagator of eq. (13) is recovered. Since $\gamma'_5 = -\gamma_5$, we conclude that the modes corresponding to the two Fermi points have, as expected, opposite chirality. Note that the symmetry $ap_4 \rightarrow \pi - ap_4$ exchanges the zeros.

In analogy to Boriçi-Creutz fermions, we have also derived the quark-quark-gluon vertex for Karsten-Wilczek fermions, which reads

$$V_1(p_1, p_2) = -ig_0 \left(\gamma_{\mu} \cos \frac{a(p_1 + p_2)_{\mu}}{2} + \gamma_4 (1 - \delta_{\mu 4}) \sin \frac{a(p_1 + p_2)_{\mu}}{2} \right), \quad (15)$$

while the quark-quark-gluon-gluon vertex comes out as

$$V_2(p_1, p_2) = \frac{1}{2} iag_0^2 \left(\gamma_{\mu} \sin \frac{a(p_1 + p_2)_{\mu}}{2} - \gamma_4 (1 - \delta_{\mu 4}) \cos \frac{a(p_1 + p_2)_{\mu}}{2} \right). \quad (16)$$

The expressions for the vertices can also be easily derived by comparing the Dirac operator of Karsten and Wilczek with the Wilson-Dirac operator

$$D_w(p) = \frac{1}{a} \sum_{\mu} \left\{ i\gamma_{\mu} \sin ap_{\mu} + r(1 - \cos ap_{\mu}) \right\} + m_0, \quad (17)$$

and observing that the hopping terms of these two actions are related by the replacement $r \leftrightarrow i\gamma_4$, with a further restriction of the term proportional to γ_4 to its spatial components only.

3 Counterterms

A key result of this paper is the observation that the actions discussed in the previous section do not contain all possible allowed operators that are invariant under the subgroup of the hyper-cubic group which is left as a symmetry (preserving the positive major diagonal for the Boriçi-Creutz case or the temporal axis for Karsten-Wilczek). We will describe in detail how radiative corrections generate new contributions whose form is not matched by any of the terms in the original bare actions. These operators must then be added as counterterms in order to construct a consistent theory under renormalization, and this consistency requirement will uniquely determine the magnitude of their coefficients.

In the following we will consider the massless case, $m_0 = 0$. Our task is to construct and add to the bare actions presented in Section 2 all possible counterterms which are allowed by the remaining symmetries. For this purpose we consider operators of dimension four or lower. We first classify them using the common notation in the continuum and then proceed to specify convenient lattice representations of these operators.

We begin with the fermionic part of the actions. The presence of a conventional chiral symmetry strongly restricts the number of possible counterterms. Since they have to anti-commute with γ_5 , we can restrict the list to those operators which contain γ_{μ} . Other Dirac matrices like 1 , γ_5 , $\gamma_{\mu}\gamma_5$ and $\sigma_{\mu\nu}$ can be excluded. The particular way in which symmetry breaking occurs in Boriçi-Creutz fermions implies that we are allowed to construct operators where summations over single indices occur, in addition to the standard Einstein summation over two indices. Then also operators containing $\sum_{\mu} \gamma_{\mu} = \Gamma$ are permitted.

As a consequence, for Boriçi-Creutz fermions there is only one possible counterterm of dimension four, namely $\bar{\psi} \Gamma \sum_{\mu} D_{\mu} \psi$ (which amounts to a renormalization of the speed of light for the fermions, relative to the positive diagonal axis). We can represent it on the lattice by writing it in a form similar to that of the hopping terms already present in the action. More precisely, we use the gauge invariant expression

$$c_4(g_0) \frac{1}{2a} \sum_{\mu} \left(\bar{\psi}(x) \Gamma U_{\mu}(x) \psi(x + a\hat{\mu}) - \bar{\psi}(x + a\hat{\mu}) \Gamma U_{\mu}^{\dagger}(x) \psi(x) \right). \quad (18)$$

There is also one counterterm of dimension three, which is constructed from Γ , i.e.

$$\frac{ic_3(g_0)}{a} \bar{\psi}(x) \Gamma \psi(x), \quad (19)$$

which is already present in the bare Boriçi-Creutz action, albeit with fixed coefficient $-2/a$. In the general renormalized action the coefficient of this operator must be tuned. It is convenient in our perturbative work to use the convention that the operator $\bar{\psi}(x)\Gamma\psi(x)$ in the renormalized action has as a coefficient $(-2 + c_3)i/a$, although for Monte Carlo simulations

other choices might be more appropriate. Since this piece of the action determines the location of the poles of the propagators, which are moved by radiative corrections, a possible renormalization condition is the requirement that the value of the coefficient must restore the poles to their original positions at p_1 and p_2 .

For Karsten-Wilczek fermions things work out in a similar way. Here we are allowed to construct objects in which Kronecker deltas can constrain any Lorentz index to be equal to 4. It is easy to see that the only gauge-invariant counterterm of dimension four that can be added to the bare action is $\bar{\psi} \gamma_4 D_4 \psi$. A suitable discretization for this operator is

$$d_4(g_0) \frac{1}{2a} \left(\bar{\psi}(x) \gamma_4 U_4(x) \psi(x + a\hat{4}) - \bar{\psi}(x + a\hat{4}) \gamma_4 U_4^\dagger(x) \psi(x) \right). \quad (20)$$

The counterterm of dimension three, i.e.

$$\frac{id_3(g_0)}{a} \bar{\psi}(x) \gamma_4 \psi(x), \quad (21)$$

is already contained in the bare Karsten-Wilczek action, where it has a fixed coefficient of $3/a$. In the fully renormalized theory we will denote the coefficient of the term $\bar{\psi}(x) \gamma_4 \psi(x)$ by $(3 + d_3) i/a$.

In perturbation theory the coefficients multiplying the above counterterms are functions of the coupling, starting at order g_0^2 . They generate new vertices and propagator insertions. At one loop they give rise to additional contributions to fermion lines, and these insertions must be taken into account for a consistent one-loop calculation. From the above expressions the rules for the corrections to external fermion propagators, which will be needed for the calculations presented in this paper, can be easily derived. These are independent of the lattice discretization chosen for the counterterms. Since the propagator is the inverse of the quadratic part of the action, they are given in momentum space by

$$-ic_4(g_0) \Gamma \sum_{\nu} p_{\nu}, \quad -\frac{ic_3(g_0)}{a} \Gamma \quad (22)$$

for Boriçi-Creutz fermions, and

$$-id_4(g_0) \gamma_4 p_4, \quad -\frac{id_3(g_0)}{a} \gamma_4 \quad (23)$$

for Karsten-Wilczek fermions, respectively. We will determine all their coefficients at one loop in perturbation theory (see Section 4), by requiring that the renormalized self-energy assumes its standard Lorentz-invariant form.

We also need counterterms for the pure gauge part of the actions of minimally doubled fermions. Although at the bare level the breaking of hyper-cubic symmetry is generated by the fermionic actions only, it propagates via the interactions between quarks and gluons also to the gauge sector in the renormalized theories. One effect is that some of the terms in the purely gluonic part can renormalize with different factors, and as a consequence pure gauge counterterms must be added to the renormalized actions to correct this imbalance. They are of the (continuum) $\text{Tr } FF$ form, but with non-conventional choices of the indices which reflect the breaking of the Lorentz (hyper-cubic) symmetry.

Let us consider first the Boriçi-Creutz case. If we choose all four indices of $\text{Tr } FF$ to appear only once in each summation (as allowed by hyper-cubic symmetry breaking), we can construct a counterterm which has the continuum form

$$\sum_{\lambda, \rho, \sigma, \tau} \text{Tr } F_{\lambda\rho}(x) F_{\sigma\tau}(x). \quad (24)$$

However, since $F_{\mu\nu}$ is antisymmetric in the indices, this is identically zero. The next possibility is to contract one Lorentz index shared by both field tensors (as in the usual Einstein convention), whilst summing individually over the two remaining indices, i.e.

$$c_P(g_0) \sum_{\lambda, \rho, \tau} \text{Tr} F_{\lambda\rho}(x) F_{\rho\tau}(x). \quad (25)$$

This operator, whose form is reminiscent of the energy-momentum tensor, is the only possible purely gluonic counterterm for this action.⁵ A lattice counterpart for this counterterm can be obtained by employing the widely used “clover” expression of the $F_{\mu\nu}$ tensor [18].

At one loop this counterterm contributes only via insertions in gluon propagators. Denoting the fixed external indices at both ends of these lines with μ and ν , all possible lattice discretizations of this counterterm yield the same Feynman rule in momentum space, namely⁶

$$-c_P(g_0) \left[(p_\mu + p_\nu) \sum_\lambda p_\lambda - p^2 - \delta_{\mu\nu} \left(\sum_\lambda p_\lambda \right)^2 \right]. \quad (28)$$

As we will see explicitly in Section 6, the presence of this counterterm is essential in order to ensure the correct renormalization of the vacuum polarization.

It is not hard to infer that in the case of Karsten-Wilczek fermions the temporal plaquettes (the chromo-electric field) renormalize differently compared with the spatial plaquettes (corresponding to the chromo-magnetic field). The counterterm to be introduced will contain an asymmetry between these two kinds of plaquettes, and can be written in continuum form as

$$d_P(g_0) \sum_{\rho, \lambda} \text{Tr} F_{\rho\lambda}(x) F_{\rho\lambda}(x) \delta_{\rho 4}. \quad (29)$$

This is the only required gluonic counterterm for this action, since introducing another factor of $\delta_{\lambda 4}$ in the above expression will produce a vanishing object. It is immediate to write down a lattice representation for it, using the plaquette:

$$d_P(g_0) \frac{\beta}{2} \sum_{\rho, \lambda} \left(1 - \frac{1}{N_C} \text{Tr} P_{4\lambda}(x) \right). \quad (30)$$

The Feynman rule for an external gluon line that this counterterm generates reads

$$-d_P(g_0) \left[p_\mu p_\nu (\delta_{\mu 4} + \delta_{\nu 4}) - \delta_{\mu\nu} (p^2 \delta_{\mu 4} \delta_{\nu 4} + p_4^2) \right]. \quad (31)$$

This term guarantees the correct renormalization of the vacuum polarization.

From the gauge-invariant expressions of the operators introduced in this section we can read off that interaction vertices are generated by the counterterms. However, these vertex

⁵In fact, choosing two pairs of summed indices will reproduce the usual Lorentz invariant term $\sum_{\lambda, \rho} \text{Tr} F_{\lambda\rho}(x) F_{\lambda\rho}(x)$, already present in the bare action.

⁶The lowest order in the expansion of $F_{\lambda\rho}(x) F_{\rho\tau}(x)$ in momentum space is

$$-p_\lambda p_\rho A_\rho A_\tau - p_\tau p_\rho A_\rho A_\lambda + p_\rho^2 A_\lambda A_\tau + p_\lambda p_\tau A_\rho A_\rho. \quad (26)$$

When the external indices of the gluon propagator are set equal to μ and ν , this becomes

$$-p_\mu p_\lambda A_\mu A_\nu - p_\nu p_\lambda A_\mu A_\nu + p^2 A_\mu A_\nu + p_\lambda p_\tau A_\mu A_\nu \delta_{\mu\nu}. \quad (27)$$

Here all indices but μ and ν are summed. A similar derivation holds for the Karsten-Wilczek action, where it gives eq. (31).

insertions are of higher order in g_0 (at least of $O(g_0^3)$), and thus they cannot contribute to the one-loop amplitudes presented in this paper.

It is worth stressing that the form of the counterterms that we have constructed remains the same at all orders of perturbation theory. Only the values of the coefficients change according to the loop order considered. The same counterterms appear also at the non-perturbative level, and will be required for a consistent numerical simulation of these fermions, as we will see in Section 7. Finally, we want to emphasize here that counterterms not only provide additional Feynman rules for the calculation of loop amplitudes, but can also modify Ward identities and hence, in particular, contribute additional terms to the conserved currents, as we will see explicitly in Section 5.

4 Fermionic counterterms at one loop

We now show how the coefficients of the counterterms that appear in the quark action can be fixed by computing the quark self-energy. Technical details in the case of Karsten-Wilczek fermions are deferred to appendix A.1, while the case of Boriçi-Creutz action was described at length in [11].

In the following we will work in some general covariant gauge, where $\partial_\mu A_\mu = 0$ and α denotes the gauge parameter. For Boriçi-Creutz fermions, the result for the one-loop diagrams of the quark self-energy, without including the counterterms, reads (see eq. (15) of [11])

$$\Sigma(p, m_0) = i\not{p} \Sigma_1(p) + m_0 \Sigma_2(p) + c_1(g_0) \cdot i \Gamma \sum_\mu p_\mu + c_2(g_0) \cdot i \frac{\Gamma}{a}, \quad (32)$$

where

$$\Sigma_1(p) = \frac{g_0^2}{16\pi^2} C_F \left[\log a^2 p^2 + 6.80663 + (1 - \alpha) \left(-\log a^2 p^2 + 4.792010 \right) \right], \quad (33)$$

$$\Sigma_2(p) = \frac{g_0^2}{16\pi^2} C_F \left[4 \log a^2 p^2 - 29.48729 + (1 - \alpha) \left(-\log a^2 p^2 + 5.792010 \right) \right], \quad (34)$$

$$c_1(g_0) = 1.52766 \cdot \frac{g_0^2}{16\pi^2} C_F, \quad (35)$$

$$c_2(g_0) = 29.54170 \cdot \frac{g_0^2}{16\pi^2} C_F, \quad (36)$$

with $C_F = (N_c^2 - 1)/2N_c$. From eq. (32) alone, the inverse propagator at one loop level would assume the form

$$\Sigma^{-1}(p, m_0) = (1 - \Sigma_1) \cdot \left\{ i\not{p} + m_0 (1 - \Sigma_2 + \Sigma_1) - i c_1(g_0) \Gamma \sum_\mu p_\mu - \frac{i c_2(g_0)}{a} \Gamma \right\}. \quad (37)$$

We now show how the coefficients multiplying the counterterms of the Boriçi-Creutz quark action can be fixed by imposing the condition that the renormalized propagator take the standard form

$$\Sigma(p, m_0) = \frac{Z_2}{i\not{p} + Z_m m_0}, \quad (38)$$

where the wave-function and quark mass renormalization factors are given by

$$Z_2 = (1 - \Sigma_1)^{-1}, \quad (39)$$

$$Z_m = 1 - (\Sigma_2 - \Sigma_1), \quad (40)$$

Indeed, by looking at eq.(22) we see that the piece in eq.(37) which is proportional to $c_1(g_0)$ can be eliminated by tuning the coefficient of the counterterm of dimension four, $c_4(g_0) \bar{\psi} \Gamma \sum_{\mu} D_{\mu} \psi$, while the one proportional to $c_2(g_0)$ can be eliminated by a suitable choice of the counterterm of dimension three, $ic_3(g_0) \bar{\psi} \Gamma \psi / a$. After inserting these counterterms, one has

$$\begin{aligned} \frac{1}{i\not{p} + m_0} &+ \frac{1}{i\not{p} + m_0} \cdot \left[i\not{p} \Sigma_1 + m_0 \Sigma_2 + c_1 \cdot i \Gamma \sum_{\mu} p_{\mu} + c_2 \cdot i \frac{\Gamma}{a} - c_4 \cdot i \Gamma \sum_{\mu} p_{\mu} \right. \\ &\quad \left. - c_3 \cdot i \frac{\Gamma}{a} \right] \cdot \frac{1}{i\not{p} + m_0} \\ &= \frac{1}{i\not{p} (1 - \Sigma_1) + m_0 (1 - \Sigma_2) - (c_1 - c_4) \cdot i \Gamma \sum_{\mu} p_{\mu} - (c_2 - c_3) \cdot i \frac{\Gamma}{a}}. \end{aligned} \quad (41)$$

It follows that $c_4(g_0) = c_1(g_0)$ and $c_3(g_0) = c_2(g_0)$, if eqs. (38)–(40) are to be recovered. Thus, at this order the coefficients of these counterterms for Boriçi-Creutz fermions are determined as

$$c_3(g_0) = 29.54170 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4), \quad (42)$$

$$c_4(g_0) = 1.52766 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4). \quad (43)$$

Of course, in order to be useful for numerical simulations, these coefficients will have to be determined non-perturbatively, by imposing suitable renormalization conditions. We will return to this question in Section 7.

The explicit calculation of the self-energy for Karsten-Wilczek fermions at one loop proceeds along the same lines as for Boriçi-Creutz. Further details as well as the individual results for the tadpole and sunset diagrams can be found in Appendix A.1. The result for the total contribution, without counterterms, is

$$\Sigma(p, m_0) = i\not{p} \Sigma_1(p) + m_0 \Sigma_2(p) + d_1(g_0) \cdot i \gamma_4 p_4 + d_2(g_0) \cdot i \frac{\gamma_4}{a}, \quad (44)$$

where

$$\Sigma_1(p) = \frac{g_0^2}{16\pi^2} C_F \left[\log a^2 p^2 + 9.24089 + (1 - \alpha) \left(-\log a^2 p^2 + 4.792010 \right) \right], \quad (45)$$

$$\Sigma_2(p) = \frac{g_0^2}{16\pi^2} C_F \left[4 \log a^2 p^2 - 24.36875 + (1 - \alpha) \left(-\log a^2 p^2 + 5.792010 \right) \right], \quad (46)$$

$$d_1(g_0) = -0.12554 \cdot \frac{g_0^2}{16\pi^2} C_F, \quad (47)$$

$$d_2(g_0) = -29.53230 \cdot \frac{g_0^2}{16\pi^2} C_F. \quad (48)$$

Without any counterterms, the inverse propagator at one loop is

$$\Sigma^{-1}(p, m_0) = \left(1 - \Sigma_1 \right) \cdot \left(i\not{p} + m_0 \left(1 - \Sigma_2 + \Sigma_1 \right) - id_1 \gamma_4 p_4 - \frac{id_2}{a} \gamma_4 \right). \quad (49)$$

As in the case of Boriçi-Creutz fermions, the extra contributions proportional to $d_1(g_0)$ and $d_2(g_0)$ can be cancelled by suitably tuning the coefficients of the counterterms $d_4(g_0) \bar{\psi} \gamma_4 D_4 \psi$

and $id_3(g_0)\bar{\psi}\gamma_4\psi/a$ respectively. Thus, the full inverse propagator at one loop for Karsten-Wilczek fermions can be written in the standard form, eqs. (38)–(40). The coefficients of the counterterms for Karsten-Wilczek fermions so determined are at one-loop order

$$d_3(g_0) = -29.53230 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4), \quad (50)$$

$$d_4(g_0) = -0.12554 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4). \quad (51)$$

One may expect that the above subtraction procedure can be carried out consistently at every order of perturbation theory. After the subtractions via the appropriate counterterms are properly taken into account, the extra terms appearing in the self-energy can be eliminated.

Recalling that the term proportional to γ_4 in the Karsten-Wilczek action can be multiplied by a parameter λ , one may wonder whether a suitable choice of λ could eliminate the power-divergent contribution to the quark self-energy, without resorting to counterterms. This could in principle be accomplished in view of the fact that the tadpole and sunset diagrams contribute with opposite sign to the self-energy, while the tadpole contribution is linearly proportional to λ . We have investigated this issue using our perturbative expressions, and concluded, however, that such a cancellation cannot take place. The reason is that while the value of the tadpole decreases as λ is lowered, the contribution of the sunset diagram also decreases at the same time. Since the latter diagram always remains much smaller than the tadpole it cannot compensate its value. Thus, it is not possible to eliminate this power-divergent extra term without counterterms.

5 Quark bilinears and conserved currents

For many applications, knowledge of the renormalization factors for quark bilinears is required. These are obtained by computing the appropriate vertex diagrams and adding the contribution Σ_1 to the quark self-energy. For Boriçi-Creutz fermions, the results for the vertex diagrams of the various bilinears are given in [11]. For Karsten-Wilczek fermions the corresponding results are listed in Appendix A.2. Here we note that for the latter, hyper-cubic symmetry breaking induces different radiative corrections for temporal and spatial components of the vector and axial-vector currents.

In the following we discuss the issue of the renormalization of the quark mass. For both realizations of minimally doubled fermions, chiral symmetry is preserved and so the quark mass does not undergo any additive renormalization. Thus, the relation between the bare and renormalized quark masses is

$$m_R = Z_m m_0, \quad (52)$$

where Z_m is given in eq. (40). The full expression for the renormalization factors of the scalar and pseudo-scalar densities in perturbation theory at one loop is

$$Z_S = Z_P = 1 - (\Lambda_S + \Sigma_1), \quad (53)$$

where Λ_S is the vertex correction of the scalar density, which is given by eq. (82) for Karsten-Wilczek and eq. (25) of [11] for Boriçi-Creutz fermions, respectively. This number is exactly equal to the $O(g_0^2)$ -contribution to Σ_2 (eq. (34) or (46)), but comes with an opposite sign: $\Lambda_S = -\Sigma_2$. Thus, when we compare them with eq. (40), we see that the renormalization factors Z_S and Z_P of the scalar and pseudo-scalar densities satisfy

$$1/Z_m = Z_S = Z_P, \quad (54)$$

where the last equality is a consequence of chiral symmetry. We have thus verified at one loop that the renormalization of the quark mass for both variants of minimally doubled fermions presented here has the same form as, say, in the case of overlap or staggered fermions.

Using the expressions for the vertex corrections of the local vector and axial-vector currents and taking the renormalization of the wave-function into account, the corresponding renormalization factors Z_V and Z_A are not equal to one. In order to construct the conserved currents, which are protected against renormalization, one has to derive the chiral Ward identities, for instance, along the lines of ref. [19]. To do this, we have to use the expressions for the lattice actions of Boriçi-Creutz and Karsten-Wilczek fermions in position space, which are given in eqs. (1) and (3) respectively.

It is important to note that the counterterms which have been added to the actions can also contribute to the chiral Ward identities, and thus provide new terms in the expressions of the conserved currents. Actually, it is easy to see that the counterterms of dimension three do not modify the Ward identities, but the counterterms of dimension four do. The latter generate additional terms in the Ward identities and hence also in the conserved currents. Thus, the form of the conserved currents turns out to be different from what the bare actions would have given.

It is worth stressing again that, as we already discussed in [11], these actions in the massless case are invariant under a chiral $U(1) \otimes U(1)$ transformation. The chiral Ward identities associated with these exact symmetries yield the currents of the theory. The chiral $U(1) \otimes U(1)$ symmetry of minimally doubled fermions implies that as the quark mass is tuned to zero there is only one Goldstone boson, which can be naturally considered to be the neutral pion. The charged pions will instead be massive in the chiral limit (at non-zero lattice spacing).

If one applies the standard vector and axial transformations, i.e.

$$\begin{aligned}\delta_V \psi &= i\alpha_V \psi, & \delta_V \bar{\psi} &= -i\bar{\psi}\alpha_V, \\ \delta_A \psi &= i\alpha_A \gamma_5 \psi, & \delta_A \bar{\psi} &= i\bar{\psi}\alpha_A \gamma_5,\end{aligned}\tag{55}$$

under which the Lagrangian remains invariant, one can identify the conserved vector current for Boriçi-Creutz fermions in the renormalized theory as

$$\begin{aligned}V_\mu^c(x) &= \frac{1}{2} \left(\bar{\psi}(x) (\gamma_\mu + i\gamma'_\mu) U_\mu(x) \psi(x + a\hat{\mu}) + \bar{\psi}(x + a\hat{\mu}) (\gamma_\mu - i\gamma'_\mu) U_\mu^\dagger(x) \psi(x) \right) \\ &\quad + \frac{c_4(g_0)}{2} \left(\bar{\psi}(x) \Gamma U_\mu(x) \psi(x + a\hat{\mu}) + \bar{\psi}(x + a\hat{\mu}) \Gamma U_\mu^\dagger(x) \psi(x) \right).\end{aligned}\tag{56}$$

The axial-vector current (which is only conserved in the massless case) is given by

$$\begin{aligned}A_\mu^c(x) &= \frac{1}{2} \left(\bar{\psi}(x) (\gamma_\mu + i\gamma'_\mu) \gamma_5 U_\mu(x) \psi(x + a\hat{\mu}) + \bar{\psi}(x + a\hat{\mu}) (\gamma_\mu - i\gamma'_\mu) \gamma_5 U_\mu^\dagger(x) \psi(x) \right) \\ &\quad + \frac{c_4(g_0)}{2} \left(\bar{\psi}(x) \Gamma \gamma_5 U_\mu(x) \psi(x + a\hat{\mu}) + \bar{\psi}(x + a\hat{\mu}) \Gamma \gamma_5 U_\mu^\dagger(x) \psi(x) \right).\end{aligned}\tag{57}$$

For Karsten-Wilczek fermions the conserved vector current turns out to be

$$\begin{aligned}V_\mu^c(x) &= \frac{1}{2} \left(\bar{\psi}(x) (\gamma_\mu - i\gamma_4 (1 - \delta_{\mu 4})) U_\mu(x) \psi(x + a\hat{\mu}) \right. \\ &\quad \left. + \bar{\psi}(x + a\hat{\mu}) (\gamma_\mu + i\gamma_4 (1 - \delta_{\mu 4})) U_\mu^\dagger(x) \psi(x) \right) \\ &\quad + \frac{d_4(g_0)}{2} \left(\bar{\psi}(x) \gamma_4 U_4(x) \psi(x + a\hat{4}) + \bar{\psi}(x + a\hat{4}) \gamma_4 U_4^\dagger(x) \psi(x) \right),\end{aligned}\tag{58}$$

and the axial-vector current is given by

$$\begin{aligned}
A_\mu^c(x) = & \frac{1}{2} \left(\bar{\psi}(x) (\gamma_\mu - i\gamma_4 (1 - \delta_{\mu 4})) \gamma_5 U_\mu(x) \psi(x + a\hat{\mu}) \right. \\
& \left. + \bar{\psi}(x + a\hat{\mu}) (\gamma_\mu + i\gamma_4 (1 - \delta_{\mu 4})) \gamma_5 U_\mu^\dagger(x) \psi(x) \right) \\
& + \frac{d_4(g_0)}{2} \left(\bar{\psi}(x) \gamma_4 \gamma_5 U_4(x) \psi(x + a\hat{4}) + \bar{\psi}(x + a\hat{4}) \gamma_4 \gamma_5 U_4^\dagger(x) \psi(x) \right). \quad (59)
\end{aligned}$$

It is instructive to verify that the renormalization constants of these currents is equal to one, since this exercise illustrates the important rôle of the counterterms. Here we restrict a detailed discussion to the conserved vector current for Karsten-Wilczek fermions, noting that this is entirely analogous to the Boriçi-Creutz case. Furthermore, the corresponding expressions for the conserved axial-vector current in both discretizations are trivially obtained from the formulae below by replacing γ_μ by $\gamma_\mu \gamma_5$ and γ_4 with $\gamma_4 \gamma_5$.

The renormalization factor of the vector current is given by

$$Z_V = 1 - (\Lambda_V + \Sigma_1), \quad (60)$$

where Λ_V denotes the vertex correction and Σ_1 the self-energy. The sum of the vertex (diagram (a) in Fig. 1), the “sails” (diagrams (b) and (c)) and the operator tadpole (diagram (d)) corresponds to the first two lines in eq. (58) and yields

$$\frac{g_0^2}{16\pi^2} C_F \gamma_\mu \left[-\log a^2 p^2 - 9.24089 + \delta_{\mu 4} \cdot 0.12554 + (1 - \alpha) \left(\log a^2 p^2 - 4.79201 \right) \right]. \quad (61)$$

Now, a counterterm is also included in the expression of the conserved current (c.f. the last line of eq. (58)), and contributes a factor of

$$d_4(g_0) \gamma_4 = d_4(g_0) \gamma_\mu \delta_{\mu 4} \quad (62)$$

to its renormalization factor. At lowest order in perturbation theory this was evaluated as part of the determination of the self-energy, with the result

$$d_4(g_0) = -0.12554 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4). \quad (63)$$

This cancels exactly the contribution proportional to $\delta_{\mu 4}$ in eq. (61). So, apart from including the wave-function renormalization, the result for the proper diagrams is

$$\Lambda_V = \frac{g_0^2}{16\pi^2} C_F \gamma_\mu \left[-\log a^2 p^2 - 9.24089 + (1 - \alpha) \left(\log a^2 p^2 - 4.79201 \right) \right]. \quad (64)$$

Finally, we see that this expression is equal and opposite to the contribution of $\Sigma_1(p)$ of the quark self-energy, eq. (45), and so exactly compensates it. Thus, we conclude that the renormalization factor of these point-split currents is one. This holds to all significant digits that we have achieved, and confirms that the expressions that we have derived via the chiral Ward identities are indeed conserved currents.

For Boriçi-Creutz fermions, the sum of vertex, sails and operator tadpole diagrams for the conserved vector current is [11]

$$\frac{g_0^2}{16\pi^2} C_F \gamma_\mu \left[-\log a^2 p^2 - 6.80664 + (1 - \alpha) \left(\log a^2 p^2 - 4.79201 \right) \right] - 1.52766 \cdot \frac{g_0^2}{16\pi^2} C_F \cdot \Gamma. \quad (65)$$

Again, at this point one has to add the contribution from the counterterm of dimension four in the conserved current (i.e. the last line of eq. (56)),

$$c_4(g_0) \Gamma, \quad (66)$$

whose coefficient was already fixed by the result of the one-loop self-energy:

$$c_4(g_0) = 1.52766 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4). \quad (67)$$

This cancels the Lorentz non-invariant term in eq. (65). After including the contribution of the wave-function renormalization it can be easily seen that the renormalization constant of this current is equal to one as well.

6 Vacuum polarization

In order to gain a deeper understanding of the properties of minimally doubled fermions, we have also taken the task to calculate the one-loop vacuum polarization of the gluon for our two realizations of minimally doubled fermions.

Here we focus on the radiative corrections to the bare gluon propagator which arise from fermion loops and the gluonic counterterm. At one loop the perturbative contributions to the vacuum polarization due to loops of gluons and ghosts are independent of the chosen fermionic lattice action, and are thus irrelevant for the problem we are studying. However, quark loops are able to generate hyper-cubic-breaking terms, and here we show that this is what indeed happens for both Karsten-Wilczek and Boriçi-Creutz fermions.

It is instructive to recall the expression for one flavour of Wilson fermions. In this case, neither breaking of hyper-cubic symmetry nor fermion doubling take place, and the result is

$$\Pi_{\mu\nu}^{(f)}(p) = \left(p_\mu p_\nu - \delta_{\mu\nu} p^2 \right) \left[\frac{g_0^2}{16\pi^2} C_2 \left(-\frac{4}{3} \log p^2 a^2 + 4.337002 \right) \right], \quad (68)$$

where C_2 is defined via $\text{Tr}(t^a t^b) = C_2 \delta^{ab}$. It is straightforward to see that this gauge invariant result satisfies the Ward identity $p^\mu \Pi_{\mu\nu}^{(f)}(p) = 0$, which expresses the conservation of the fermionic current.

The result of our calculations for the above quantity using Boriçi-Creutz fermions, without including the gluonic counterterm, reads

$$\begin{aligned} \Pi_{\mu\nu}^{(f)}(p) &= \left(p_\mu p_\nu - \delta_{\mu\nu} p^2 \right) \left[\frac{g_0^2}{16\pi^2} C_2 \left(-\frac{8}{3} \log p^2 a^2 + 23.6793 \right) \right] \\ &\quad - \left((p_\mu + p_\nu) \sum_\lambda p_\lambda - p^2 - \delta_{\mu\nu} \left(\sum_\lambda p_\lambda \right)^2 \right) \frac{g_0^2}{16\pi^2} C_2 \cdot 0.9094. \end{aligned} \quad (69)$$

By comparing the coefficient of the divergence to the expression for Wilson fermions we see explicitly that this result corresponds to two flavours of quarks: Each of the two doublers contributes an equal amount to the divergence. The same is true for Karsten-Wilczek fermions, for which our calculation gives the result (again without counterterms)

$$\begin{aligned} \Pi_{\mu\nu}^{(f)}(p) &= \left(p_\mu p_\nu - \delta_{\mu\nu} p^2 \right) \left[\frac{g_0^2}{16\pi^2} C_2 \left(-\frac{8}{3} \log p^2 a^2 + 19.99468 \right) \right] \\ &\quad - \left(p_\mu p_\nu (\delta_{\mu 4} + \delta_{\nu 4}) - \delta_{\mu\nu} (p^2 \delta_{\mu 4} \delta_{\nu 4} + p_4^2) \right) \frac{g_0^2}{16\pi^2} C_2 \cdot 12.69766. \end{aligned} \quad (70)$$

We first note, looking at the second lines of the above equations, that additional terms appear, compared with a standard situation like Wilson fermions. A remarkable observation is the fact that although each of the two actions breaks hyper-cubic symmetry, the extra terms still satisfy the Ward identity $p^\mu \Pi_{\mu\nu}^{(f)}(p) = 0$, expressing current conservation, as can be easily verified. It is easy to see that one cannot obtain smaller expressions for the symmetry-breaking terms which are symmetric in μ and ν and still conserve the current. Moreover, our one loop results are the most general symmetric quadratic functions of p which can be constructed using the hyper-cubic breaking objects $\delta_{\mu 4}$ or $\sum_\lambda p_\lambda$.

The correct renormalization of the polarization of the vacuum requires the inclusion of the counterterm of the pure gauge action. To see how this works in detail, we consider for illustration Boriçi-Creutz fermions. Adding the one-loop results to the tree-level expression, and exploiting $p^\mu \Pi_{\mu\nu}^{(f)}(p) = 0$ in the usual way, we can write

$$\begin{aligned} & \frac{\delta_{\mu\nu} - \alpha \frac{p_\mu p_\nu}{p^2}}{p^2} + \frac{\delta_{\mu\lambda} - \alpha \frac{p_\mu p_\lambda}{p^2}}{p^2} \cdot \left[(p_\lambda p_\rho - \delta_{\lambda\rho} p^2) \Pi(p^2) \right. \\ & \quad \left. + \left((p_\lambda + p_\rho) \sum_\tau p_\tau - p^2 - \delta_{\lambda\rho} \left(\sum_\tau p_\tau \right)^2 \right) \tilde{\Pi}(p^2) \right] \cdot \frac{\delta_{\rho\nu} - \alpha \frac{p_\rho p_\nu}{p^2}}{p^2} \\ & = \frac{\delta_{\mu\nu} - \tilde{\alpha} \frac{p_\mu p_\nu}{p^2}}{p^2(1 - \Pi(p^2))} + \frac{(p_\mu + p_\nu) \sum_\tau p_\tau - p^2 - \delta_{\mu\nu} \left(\sum_\tau p_\tau \right)^2}{p^4} \tilde{\Pi}(p^2). \end{aligned} \quad (71)$$

The function $\Pi(p^2)$ in the first line of the above equation multiplies the standard Lorentz invariant one-loop expression, which is then rearranged such as to produce the functional form of the continuum tree-level gluon propagator in the last line. From this one can read off that the gauge parameter is renormalized according to $\tilde{\alpha} = \alpha(1 - \Pi) + \Pi$, and that $Z_3 = 1/(1 - \Pi(0))$. The remaining terms, the ones proportional to $\tilde{\Pi}(p^2)$, are those that break hyper-cubic symmetry, and they cannot be rearranged in a similar way.

It is thus evident that these hyper-cubic-breaking contributions must be eliminated, and this can be achieved by employing the gluonic counterterms which we introduced in Section 3. Indeed, the expression for the gluonic counterterm for Boriçi-Creutz fermions in momentum space, eq. (28), is structurally identical to the additional terms in the vacuum polarization. Requiring the one-loop vacuum polarization to assume the standard Lorentz invariant form (that is, only the first term in the above result) then uniquely determines the coefficient of the counterterm. The non-standard contributions are cancelled for Boriçi-Creutz fermions if we set

$$c_P(g_0) = -0.9094 \cdot \frac{g_0^2}{16\pi^2} C_2 + O(g_0^4). \quad (72)$$

The reasoning for Karsten-Wilczek fermions is entirely analogous, and one concludes that

$$d_P(g_0) = -12.69766 \cdot \frac{g_0^2}{16\pi^2} C_2 + O(g_0^4). \quad (73)$$

The most important thing to realize is that there are no power-divergences in our results for the vacuum polarization. In principle such divergences could arise with coefficients proportional to $1/a^2$ or $1/a$. We have explicitly checked in our calculations that the $1/a^2$ tadpole contribution, when non-zero, is in all cases of equal magnitude and opposite sign with respect to the sunset diagram.⁷

⁷It is interesting to note that the numbers for these diagrams are much larger than in the case of Wilson fermions, where the coefficient of $g_0^2 C_2 / 16\pi^2$ for the tadpole is -9.67590 . For Karsten-Wilczek fermions this number turns out to be -36.31464 for each spatial component and 7.12931 for the temporal component. For Boriçi-Creutz fermions it is even larger, -73.71980 .

We can understand on general grounds why such power-divergences cannot appear. To construct hyper-cubic breaking terms one has to employ objects like Γ and $\sum_\mu p_\mu$ (for Boriçi-Creutz fermions) and γ_4 and p_4 (for Karsten-Wilczek fermions). However, after the traces of the fermions loops are evaluated no Dirac structures are left over, and momenta cannot anyway appear at the $1/a^2$ level. Linear pieces in the momenta, which would be required in case of a $1/a$ power divergence, are instead prohibited by the symmetry of the diagrams.

We have also discovered that the hyper-cubic-breaking terms can be put for both actions in the same algebraic form, namely

$$p^2 \{\gamma_\mu, \Gamma\} \{\gamma_\nu, \Gamma\} + \delta_{\mu\nu} \{\not{p}, \Gamma\} \{\not{p}, \Gamma\} - \frac{1}{2} \{\not{p}, \Gamma\} \left(\{\gamma_\mu, \not{p}\} \{\gamma_\nu, \Gamma\} + \{\gamma_\nu, \not{p}\} \{\gamma_\mu, \Gamma\} \right), \quad (74)$$

where in the case of Karsten-Wilczek fermions Γ must be replaced by $\gamma_4/2$. This substitution is suggested by comparison of the standard relation $\Gamma = \frac{1}{4} \sum_\mu (\gamma_\mu + \gamma'_\mu)$ of Boriçi-Creutz fermions with the formula $\gamma_4 = \frac{1}{2} \sum_\mu (\gamma_\mu + \gamma'_\mu)$ for Karsten-Wilczek fermions, expressing the symmetries of the action (as can be seen from Section 2, when one expands the propagator of the latter action around the second Fermi point). Whether there is any deeper significance to this structural “equivalence” of the hyper-cubic-breaking structures in the vacuum polarizations remains an open question.

7 Numerical simulations

In this section we discuss the implications of our one-loop perturbative calculations for numerical simulations of minimally doubled fermions. The first thing to note is that simulations must be based on the complete renormalized actions, including the counterterms. In position space the full expression for the Boriçi-Creutz action reads

$$\begin{aligned} S_{\text{BC}}^f = a^4 \sum_x \left\{ \frac{1}{2a} \sum_{\mu=1}^4 \left[\bar{\psi}(x) (\gamma_\mu + c_4(g_0) \Gamma + i\gamma'_\mu) U_\mu(x) \psi(x + a\hat{\mu}) \right. \right. \\ \left. \left. - \bar{\psi}(x + a\hat{\mu}) (\gamma_\mu - c_4(g_0) \Gamma - i\gamma'_\mu) U_\mu^\dagger(x) \psi(x) \right] \right. \\ \left. + \bar{\psi}(x) \left(m_0 + \tilde{c}_3(g_0) \frac{i\Gamma}{a} \right) \psi(x) \right. \\ \left. + \beta \sum_{\mu < \nu} \left(1 - \frac{1}{N_C} \text{Re Tr } P_{\mu\nu} \right) + c_P(g_0) \sum_{\mu, \nu, \rho} \text{Tr } \hat{F}_{\mu\rho}(x) \hat{F}_{\rho\nu}(x) \right\}, \end{aligned} \quad (75)$$

where we have redefined the coefficient of the dimension-three counterterm, via $\tilde{c}_3(g_0) = -2 + c_3(g_0)$, and \hat{F} is some lattice discretization of the field-strength tensor. The renormalized action for Karsten-Wilczek fermions reads

$$\begin{aligned} S_{\text{KW}}^f = a^4 \sum_x \left\{ \frac{1}{2a} \sum_{\mu=1}^4 \left[\bar{\psi}(x) (\gamma_\mu (1 + c_4(g_0) \delta_{\mu 4}) - i\gamma_4 (1 - \delta_{\mu 4})) U_\mu(x) \psi(x + a\hat{\mu}) \right. \right. \\ \left. \left. - \bar{\psi}(x + a\hat{\mu}) (\gamma_\mu (1 - d_4(g_0) \delta_{\mu 4}) + i\gamma_4 (1 - \delta_{\mu 4})) U_\mu^\dagger(x) \psi(x) \right] \right. \\ \left. + \bar{\psi}(x) \left(m_0 + \tilde{d}_3(g_0) \frac{i\gamma_4}{a} \right) \psi(x) \right. \\ \left. + \beta \sum_{\mu < \nu} \left(1 - \frac{1}{N_C} \text{Re Tr } P_{\mu\nu} \right) (1 + d_P(g_0) \delta_{\mu 4}) \right\}, \end{aligned} \quad (76)$$

where $\tilde{d}_3(g_0) = 3 + d_3(g_0)$. In perturbation theory the coefficients of the counterterms admit the expansions

$$\begin{aligned}\tilde{c}_3(g_0) &= -2 + c_3^{(1)} g_0^2 + c_3^{(2)} g_0^4 + \dots, \\ c_4(g_0) &= c_4^{(1)} g_0^2 + c_4^{(2)} g_0^4 + \dots, \\ c_P(g_0) &= c_P^{(1)} g_0^2 + c_P^{(2)} g_0^4 + \dots,\end{aligned}\tag{77}$$

and

$$\begin{aligned}\tilde{d}_3(g_0) &= 3 + d_3^{(1)} g_0^2 + d_3^{(2)} g_0^4 + \dots, \\ d_4(g_0) &= d_4^{(1)} g_0^2 + d_4^{(2)} g_0^4 + \dots, \\ d_P(g_0) &= d_P^{(1)} g_0^2 + d_P^{(2)} g_0^4 + \dots.\end{aligned}\tag{78}$$

In order to define the full actions at the non-perturbative level one must impose suitable renormalization conditions which fix the values of the counterterms beyond perturbation theory. Below we discuss possible scenarios how this could be achieved.

It is fairly straightforward to determine the coefficients of the counterterms of dimension four via a non-perturbative condition: as we have seen in Section 5, these counterterms ensure that the conserved currents have unit renormalization. The coefficients c_4 and d_4 can then be fixed by requiring that the electric charge be equal to one. To this end one can compute suitable ratios of three-point and two-point correlation functions, involving the expressions in eqs. (56) and (58), respectively. Adjusting the coefficients until the charge is unity fixes the values of c_4 and d_4 . It is an empirical fact that ratios of correlators are obtained with good statistical accuracy.

Furthermore, radiative corrections induce a shift of the poles of the quark propagator away from their tree-level positions. Provided that their coefficients are appropriately tuned, the counterterms of dimension three ensure that the two Fermi points can be moved back to their original locations. It is important to realize that radiative corrections, when moving the poles, do not introduce a sign problem into the Monte Carlo generation of configurations. The gauge action remains real, and the eigenvalues of the Dirac operator are always in complex conjugate pairs, making the fermion determinant always non-negative.

On the other hand, these shifts can introduce oscillations as a function of separation into some hadronic correlation functions. Such oscillations, familiar from the staggered formulation, come about since the underlying fermion field can create several different species, and these species occur in different regions of the Brillouin zone. It would be interesting to explore whether or not these oscillations could be cancelled by constructing hadronic operators spread over nearby neighbours.

It is important to remember that because the two species are of opposite chirality, the naive γ_5 matrix is physically a flavour non-singlet. The naive on-site pseudoscalar field $\bar{\psi}\gamma_5\psi$ can create only flavour non-singlet pseudoscalar states. To create the flavour-singlet pseudoscalar meson, which gets its mass from the anomaly, one needs to combine fields on nearby sites with appropriate phases.

The purely gluonic counterterm for Boriçi-Creutz fermions introduces new operators into the renormalized action, in which chromo-electric and chromo-magnetic fields enter. Since the positive diagonal of space time hypercubes is selected as special, terms involving $\sum_{\mu,\nu,\rho} F_{\mu\rho} F_{\nu\rho}$ can enter. This gives rise to combinations like $E \cdot B$, $E_1 E_2$, $B_2 B_3$ and similar. These cross terms can be removed by a diagonalization process, essentially a rotation redefining the time direction to be along the positive diagonal. However, then the coefficients of E^2 and B^2 can

differ. Effectively, the speed of light for the gluons has been renormalized. The coefficient c_p could then be fixed by tuning its value to restore the E, B symmetry. The above effects could turn out to be small, given that at tree level only the fermionic actions break hyper-cubic symmetry. It could also happen that other derived quantities are more sensitive to this coefficient and more suitable for its non-perturbative determination. In general, one can look for Ward identities and study their deviation from the standard Lorentz invariant form, as a function of c_p .

For Karsten-Wilczek fermions the purely gluonic counterterm induces an asymmetry between the plaquettes containing a temporal link relative to those involving spatial links only. Fixing the coefficient of this counterterm, d_p , could then be accomplished by computing a spatial plaquette or Wilson loop, and then equating its result to its counterpart with components in the time direction.

Eventually Monte Carlo simulations will reveal the actual amount of symmetry breaking, which could turn out to be large or small, depending on the observable considered. One important such quantity is the mass splitting of the charged pions relative to the neutral pion. Furthermore, the relative magnitude of these symmetry-breaking effects could be substantially different for Boriçi-Creutz and Karsten-Wilczek fermions. Which of these two realizations has the potential to become the preferred choice for numerical simulations, will largely depend on this issue.

So far we have not touched on the important subject of identifying the leading lattice artefacts associated with minimally doubled fermions. As we observed in connection with eqs. (7) and (8), both realizations of minimally doubled fermions contain a dimension-five operator in the bare action, and this leads one to expect the leading lattice artefacts to be of order a .⁸ Naively, one might assume that the preservation of chiral symmetry would automatically ensure $O(a)$ improvement, but here we can see that this is not always the case. In minimally doubled fermion actions there are in fact manifest $O(a)$ effects, which arise as a consequence of the breaking of hyper-cubic (and not chiral) symmetry.

This naturally leads to a discussion of the subject of $O(a)$ improvement. Since these actions are not improved from the beginning, at least one further dimension-five operator will be needed in order to cancel $O(a)$ contributions in on-shell matrix elements. Of course all possible operators of dimension five that are consistent with the symmetries must be considered. Since minimally doubled fermions respect chiral symmetry, this excludes operators like the “clover” term, which contains $\sigma_{\mu\nu}$ and therefore does not anticommute with γ_5 . We attempted a cursory analysis, but there appear to be quite a number of possible operators, even after some of them are reabsorbed into a rescaling of parameters or eliminated using the equations of motion. Just to give an example, among the possible operators for Boriçi-Creutz fermions we find $\bar{\psi} \Gamma \sum_{\mu,\nu} D_\mu D_\nu \psi$ and $\sum_{\mu,\nu,\lambda} F_{\mu\nu} D_\lambda F_{\mu\nu}$. We emphasize that such additional dimension-five operators can occur not only in the quark sector, but also in the pure gauge part. In fact, when Lorentz invariance is broken, the statement that only operators with even dimension can appear in the pure gauge action is no longer true. As this discussion shows, the issue of $O(a)$ improvement can be quite intricate for this particular type of fermionic discretizations. The task of classifying the minimal set of independent operators is an interesting problem but beyond the scope of this paper.

We close this section with some speculations. First, it is not entirely clear that we need to move the Fermi points back to the free theory positions. As long as there are still two poles

⁸This contradicts the findings of the (numerical) tree-level analysis of ref. [5], according to which cutoff effects for pion masses and decay constants were of order a^2 .

in the propagator, we have minimal doubling. Further investigations could reveal whether relaxing this condition simplifies the tuning of counterterms. Finally, if we are willing to live with an induced anisotropy, we only need to tune c_3 and c_4 roughly and then use physical observables to measure the remnant anisotropy. Then only c_P needs to be adjusted so that the anisotropy for the gluons is the same as for the fermions.

8 Conclusions

Following on the first investigation of minimally doubled fermions for Boriçi-Creutz fermions [11], we have presented further perturbative study including another realization, namely Karsten-Wilczek fermions. At the same time, we have re-interpreted our analysis of the renormalization of both these actions in the context of a solid field-theoretical framework.

Our investigations show that both Boriçi-Creutz and Karsten-Wilczek fermions are described by a fully coherent quantum field theory. Since the complete set of operators allowed by the symmetries of these fermions includes several not present in the tree-level actions, counterterms must be introduced for a consistent renormalized theory. After adding the counterterms, only a small number of new mixings arises for the matrix elements of local bilinears, none of which is power divergent. For the local vector and axial-vector currents, some finite mixing with hyper-cubic breaking operators occurs. For the scalar and pseudoscalar densities and the tensor operator, on the other hand, the structure of the mixings is as in the continuum.

We have also constructed the conserved vector and axial-vector currents for both kinds of fermions. The vector current is isospin-singlet, representing the conservation of fermion number. The axial current, however, is a non-singlet because the doubled fermions have opposite chirality. These currents have simple expressions which involve only nearest-neighbour points, and do not undergo any mixing. We have verified at one loop in perturbation theory that their renormalization constants are equal to one. One of the most attractive features of Boriçi-Creutz and Karsten-Wilczek fermions is that they belong to the very few lattice discretizations that yield a simple and essentially ultralocal expression for a conserved axial-vector current. These features could turn out to be a key advantage in numerical simulations. Furthermore, we have also calculated the polarization of the vacuum for both actions. We have proven that, in spite of the breaking of hyper-cubic symmetry, no power divergences appear for the vacuum polarization of the gluon.

In summary, we have constructed the renormalized theory up to and including $O(g_0^2)$ for the two realizations of minimally doubled fermions considered. We have discussed perturbative and non-perturbative conditions for fixing the coefficients of the three counterterms required for both realizations. We have argued that under reasonable assumptions and following the determination of these counterterms, no special features of these two realizations of minimally doubled fermions should hinder their successful Monte Carlo simulation.

Some questions merit further consideration: firstly, one should revisit the problem of formulating conditions which allow for a precise determination of the coefficients of the dimension-three counterterm at the non-perturbative level, and the same applies to the gluonic counterterms. Secondly, attempts to improve convergence towards the continuum limit must take account of the inherent hyper-cubic symmetry breaking and the induced mixing with dimension-five operators.

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A One-loop calculations for Karsten-Wilczek fermions

In this appendix we report the individual expressions at one-loop for various quantities computed using the Karsten-Wilczek action. Figure 1 lists all diagrams which are needed for the perturbative calculations presented in this article.

A.1 Self-energy

Using the expression for the vertex $V_2(p, p)$ of eq. (16), the tadpole contribution to the self energy can be easily computed. In a general covariant gauge, where $\partial_\mu A_\mu = 0$, its expression is

$$\begin{aligned} & \frac{1}{a^2} \cdot \frac{Z_0}{2} \left(1 - \frac{1}{4}(1 - \alpha)\right) \cdot i a g_0^2 C_F \left(\sum_{\mu=1}^4 \gamma_\mu a p_\mu - \sum_{k=1}^3 \gamma_4 (1 + O(a^2)) \right) \\ &= g_0^2 C_F \frac{Z_0}{2} \left(1 - \frac{1}{4}(1 - \alpha)\right) \left(i \not{p} - \frac{3i\gamma_4}{a} \right) + O(a), \end{aligned} \quad (79)$$

where Z_0 is given by [20–22]

$$Z_0 = \int_{-\pi/a}^{\pi/a} \frac{d^4 p}{(2\pi)^4} \frac{1}{\hat{p}^2} = 0.1549333 \dots = 24.466100 \frac{1}{16\pi^2}, \quad \hat{p}^2 = \frac{4}{a^2} \sum_{\mu} \sin^2 \left(\frac{a p_\mu}{2} \right), \quad (80)$$

and terms of $O(a)$ and higher can eventually be set to zero. The term proportional to $i \not{p}$ is the same as for Wilson fermions, while the other term would imply a power-divergent mixing of order $1/a$ with the dimension-3 operator $\bar{\psi} \gamma_4 \psi$, unless there be a cancellation with an analogous term coming from the contribution of the sunset diagrams to the self-energy (contained e.g. in diagram (e) of Figure 1). Here we show that no such compensation occurs.

We have computed the sunset diagram using special computer codes written in FORM [23, 24] and Mathematica, and also checked it against calculations by hand. The result is

$$\begin{aligned} \Sigma^{\text{sunset}}(p, m_0) &= i \not{p} \cdot \frac{g_0^2}{16\pi^2} C_F \left[\log a^2 p^2 - 2.99216 + (1 - \alpha) \left(-\log a^2 p^2 + 7.850272 \right) \right] \\ &+ m_0 \cdot \frac{g_0^2}{16\pi^2} C_F \left[4 \log a^2 p^2 - 24.36875 + (1 - \alpha) \left(-\log a^2 p^2 + 5.792010 \right) \right] \\ &- 0.12554 \cdot \frac{g_0^2}{16\pi^2} C_F \cdot i \gamma_4 p_4 + (7.16687 - 9.17479 (1 - \alpha)) \cdot \frac{g_0^2}{16\pi^2} C_F \cdot i \frac{\gamma_4}{a}. \end{aligned} \quad (81)$$

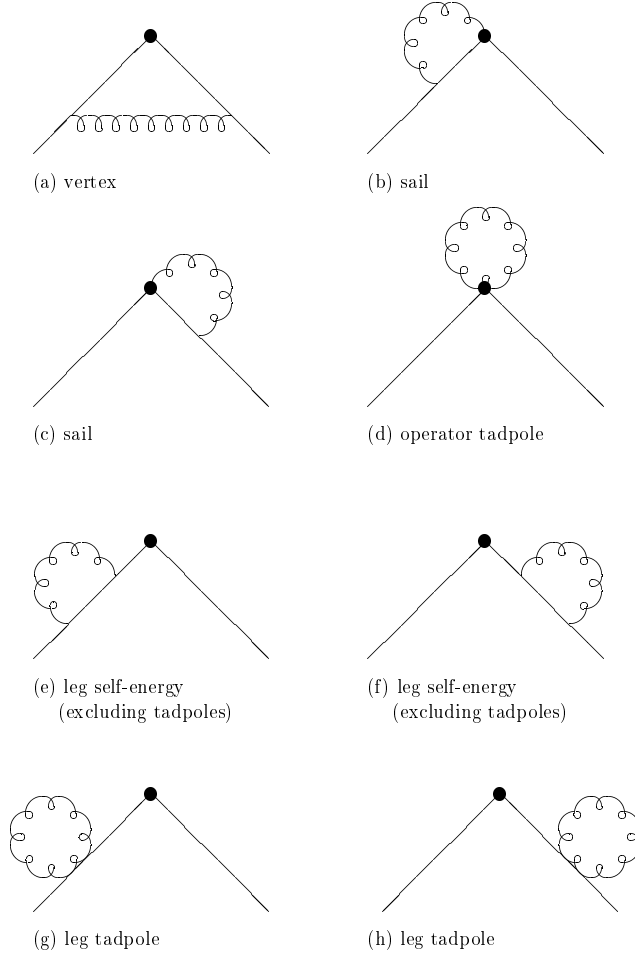


Figure 1: The diagrams needed for the one-loop renormalization of the lattice operators.

Note that gauge invariance forces the terms proportional to $(1 - \alpha)$ to be the same as, for example, in the case of Wilson or overlap fermions. This is an important check of the correctness of our calculations. The two terms proportional to γ_4/a arising from the tadpole and the sunset diagrams do not cancel – they actually reinforce each other. However, the parts proportional to $(1 - \alpha)$ cancel exactly, as required by gauge invariance.

A.2 Bilinears

Here we list the results for the individual vertex diagrams for the scalar density, as well as the vector and tensor currents. As a consequence of chiral symmetry, the vertex correction for the pseudoscalar density is identical to that of the scalar density. The same is true for the vector and axial-vector currents.

For the scalar and pseudoscalar densities the result for the vertex correction is

$$\Lambda_S = \frac{g_0^2}{16\pi^2} C_F \left[-4 \log a^2 p^2 + 24.36875 + (1 - \alpha) \left(\log a^2 p^2 - 5.792010 \right) \right]. \quad (82)$$

One can see that the breaking of hyper-cubic symmetry does not induce any mixing.

For the vector current the vertex diagram yields

$$\Lambda_V = \frac{g_0^2}{16\pi^2} C_F \gamma_\mu \left[-\log a^2 p^2 + 10.44610 - 2.88914 \cdot \delta_{\mu 4} + (1 - \alpha) \left(\log a^2 p^2 - 4.792010 \right) \right]. \quad (83)$$

This result shows explicitly that the spatial and temporal components of the vector (and also those of the axial-vector) current receive different radiative corrections. This is a consequence of the breaking of hyper-cubic symmetry, and of the special rôle taken by the temporal direction. On the other hand, it is encouraging that mixing between the spatial and temporal components appears to be absent. Each of these components still renormalizes multiplicatively, and the mixing matrix is diagonal.

Finally, for the tensor current we obtain the result for the vertex diagram as

$$\Lambda_T = \frac{g_0^2}{16\pi^2} C_F \sigma_{\mu\nu} \left[4.17551 + (1 - \alpha) \left(\log a^2 p^2 - 3.792010 \right) \right]. \quad (84)$$

Similarly to the scalar and pseudoscalar case, the breaking of hyper-cubic invariance does not generate here any extra mixing. It is remarkable that the tensor operator does not appear to show any preference for the temporal direction even after (one-loop) renormalization, that is, the renormalization constant is the same for each of the six independent components of the tensor operator.

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